

The National
**Mathematics
and Science**
College

*HOW TO SATISFY THE NEVER SATISFIED
STUDENT, OR HOW TO ACKNOWLEDGE
WHAT SHOULD BE SAID*

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DISCLAIMER:

I AM NOT AN EXPERT!

[IF THINGS GO HORRIBLY WRONG,
ASK ME ABOUT MOESSNER'S
MIRACLE... OR EVEN HULL CITY FC]

I AM HAPPY TO SHARE IDEAS IN
FUTURE... FEEL FREE TO E-MAIL ME.

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**THERE WAS A HANDOUT FOR THIS
SESSION.**

**WHAT FOLLOWS OFFERS MY
THOUGHTS ON THE HANDOUT...**

Do you derive the quadratic formula or just state it?

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

ONE PERSON SHARED THAT SHE LIKES TO JUST THROW IN “SOLVE $ax^2 + bx + c = 0$ ” AT THE END OF AN EXERCISE OF SOLVING QUADRATIC EQUATIONS BY COMPLETING THE SQUARE, AND THEN SEE THE SURPRISE ON STUDENTS’ FACES AS THEY DISCOVER THEY HAVE DERIVED THE QUADRATIC FORMULA!

THE FACTOR THEOREM

$f(x)$ IS A POLYNOMIAL FUNCTION:

$x = a$ IS A ROOT OF $f(x) = 0 \iff (x - a)$ IS A FACTOR OF $f(x)$

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IF $f(x) \equiv (x - a)g(x)$, WHERE $g(x)$ IS ALSO A POLYNOMIAL FUNCTION,
 $f(a) = 0 \times g(a) = 0$, SO $x = a$ IS A ROOT OF $f(x) = 0$

BUT $x = a$ IS A ROOT OF $f(x) = 0 \implies (x - a)$ IS A FACTOR OF $f(x)$

?

THE FACTOR THEOREM

$f(x)$ IS A POLYNOMIAL FUNCTION:

$x = a$ IS A ROOT OF $f(x) = 0 \implies (x - a)$ IS A FACTOR OF $f(x)$

?

LET $f(x) = p_0 + p_1x + p_2x^2 + p_3x^3 + \cdots + p_{n-1}x^{n-1} + p_nx^n$

$$f(a) = 0$$

SO $f(x) = f(x) - f(a)$

$$\begin{aligned} &= p_0 + p_1x + p_2x^2 + p_3x^3 + \cdots + p_{n-1}x^{n-1} + p_nx^n \\ &\quad - (p_0 + p_1a + p_2a^2 + p_3a^3 + \cdots + p_{n-1}a^{n-1} + p_na^n) \end{aligned}$$

$$= p_1(x - a) + p_2(x^2 - a^2) + p_3(x^3 - a^3) + \cdots + p_{n-1}(x^{n-1} - a^{n-1}) + p_n(x^n - a^n)$$

THE FACTOR THEOREM

$f(x)$ IS A POLYNOMIAL FUNCTION:

$x = a$ IS A ROOT OF $f(x) = 0 \implies (x - a)$ IS A FACTOR OF $f(x)$
?

BUT

$$x^2 - a^2 \equiv (x - a)(x + a)$$

$$x^3 - a^3 \equiv (x - a)(x^2 + ax + a^2)$$

$$x^4 - a^4 \equiv (x - a)(x^3 + ax^2 + a^2x + a^3)$$

...

$$x^n - a^n \equiv (x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + a^3x^{n-4} + \dots + a^{n-1}x + a^n)$$

THE FACTOR THEOREM

$f(x)$ IS A POLYNOMIAL FUNCTION:

$x = a$ IS A ROOT OF $f(x) = 0 \implies (x - a)$ IS A FACTOR OF $f(x)$

?

SO

$$\begin{aligned} f(x) &= p_1(x - a) + p_2(x^2 - a^2) + p_3(x^3 - a^3) + \cdots + p_{n-1}(x^{n-1} - a^{n-1}) + p_n(x^n - a^n) \\ &= p_1(x - a) + p_2(x - a)(x + a) + p_3(x - a)(x^2 + ax + a^2) + \cdots \\ &\quad + p_{n-1}(x - a)(x^{n-2} + ax^{n-3} + a^2x^{n-4} + \cdots + a^{n-2}) \\ &\quad + p_n(x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-1}) \\ &= (x - a)\{p_1 + p_2(x + a) + p_3(x^2 + ax + a^2) + \cdots \\ &\quad + p_{n-1}(x^{n-2} + ax^{n-3} + a^2x^{n-4} + \cdots + a^{n-2}) \\ &\quad + p_n(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-1})\} \end{aligned}$$

AND $(x - a)$ IS A FACTOR OF $f(x)$

Partial Fractions

Students just get taught the three types of partial fractions and become familiar with what to do, but there is also a lot of trust that partial fractions will always 'work'. How do we know there aren't cases where it doesn't work?

- An algebraic fraction $\frac{P(x)}{Q(x)}$, $P(x), Q(x) \neq 0$ both polynomials in x , is called proper if $\deg P(x) < \deg Q(x)$.

(Given an improper algebraic fraction, $\frac{P(x)}{Q(x)}$ with $\deg P(x) \geq \deg Q(x)$, you can always do long division of polynomials or inspection to express $\frac{P(x)}{Q(x)}$ as $R(x) + \frac{S(x)}{Q(x)}$, where $R(x)$ and $S(x)$ are polynomials with $\deg S(x) < \deg Q(x)$ [and $\deg R(x) = \deg P(x) - \deg Q(x)$])

- If $\frac{T(x)}{U(x)}$ and $\frac{V(x)}{W(x)}$ are proper algebraic fractions (so $\deg T(x) < \deg U(x)$ and $\deg V(x) < \deg W(x)$)

Partial Fractions

Students just get taught the three types of partial fractions and become familiar with what to do, but there is also a lot of trust that partial fractions will always 'work'. How do we know there aren't cases where it doesn't work?

$$\text{then } \frac{T(x)}{U(x)} + \frac{V(x)}{W(x)} = \frac{T(x)W(x) + U(x)V(x)}{U(x)W(x)}$$

- the numerator and denominator ($\neq 0$) are both clearly polynomials, since $T(x)$, $U(x) (\neq 0)$, $V(x)$ and $W(x) (\neq 0)$ are.

- $\deg(T(x)W(x) + U(x)V(x)) = \max(\deg T(x) + \deg W(x), \deg U(x) + \deg V(x))$
 $< \deg U(x) + \deg W(x)$
since $\deg T(x) < \deg U(x)$
and $\deg V(x) < \deg W(x)$.

So the sum of any two proper algebraic fractions is itself a proper algebraic fraction!

Partial Fractions

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So the sum of any two proper algebraic fractions is itself a proper algebraic fraction!

But this does not prove the converse, that if a proper algebraic fraction has a denominator that factorises then it can be expressed as a sum of two or more proper algebraic fractions.

For example,

$$\frac{2x-1}{(x+3)(x^2+1)} \equiv \frac{A}{x+3} + \frac{Bx+C}{x^2+1}$$

$$\frac{2x-1}{(x+3)(x^2+1)} \equiv \frac{A(x^2+1) + (Bx+C)(x+3)}{(x+3)(x^2+1)}$$

3 equations, 3 unknowns...

unique solution if
system is linearly independent...

Numerators:

$$[x^2]: 0 = A + B$$

$$[x]: 2 = 3B + C$$

$$[x^0]: -1 = A + 3C$$

Partial Fractions

Students just get taught the three types of partial fractions and become familiar with what to do, but there is also a lot of trust that partial fractions will always 'work'. How do we know there aren't cases where it doesn't work?

If you can always sum into proper algebraic fractions...

$$\left. \begin{array}{l} \frac{M}{x-A} \\ \frac{Nx+P}{x^2+B} \end{array} \right\} \deg(\text{num}) < \deg(\text{denom})$$

Partial Fractions

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$$\frac{Qx+R}{(x+C)^2} \equiv \frac{Q(x+C) + R-QC}{(x+C)^2} \equiv \underbrace{\frac{Q}{x+C} + \frac{R-QC}{(x+C)^2}}_{\text{simpler!}}$$

Partial Fractions

If we put this to one side, students can be unhappy about substituting $x = a$ when $(x - a)$ is a factor of the denominator of the algebraic fraction? How do you satisfy them?

I only addressed this verbally in the session.

$$\text{e.g. } \frac{x+1}{(x+2)(x-5)} \equiv \frac{A}{x+2} + \frac{B}{x-5} \equiv \frac{A(x-5) + B(x+2)}{(x+2)(x-5)}$$

Removing the denominators we really should say that $x+1$ and $A(x-5) + B(x+2)$ must agree at all values of x apart from $x = -2$ and $x = 5$.

However, we can then also say that there is a unique line between two points, so since $x+1$ and $A(x-5) + B(x+2)$ agree at a lot more than two values of $x^\dagger$, they must be the same line and so also agree at $x = -2$ and $x = 5$.

This still feels a little 'hand wavy', but it is what's going on!

$\dagger 2 \ll \infty$

Partial Fractions

SOMEONE SHARED THAT THEY LIKE TO ALLOW STUDENTS TO CONTINUE WORKING IF THEY HAVE THE WRONG STARTING POINT, THEN CHALLENGE THEM AT THE END TO CHECK THEIR ANSWERS WORK (AND TO SEE THEIR SURPRISE WHEN THEY DON'T!).

Generalised Binomial Theorem

How do you explain why $(1+x)^n$ is valid for $|x| < 1$?

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r + \dots, |x| < 1.$$

Yes, of course, if $|x| < 1$ then $x^r \rightarrow 0$ as $r \rightarrow \infty$ BUT

- $\frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$ is less obvious AND

- An infinite sum of terms that converge to 0 can diverge to infinity.
I'm thinking of $\sum_{r=1}^{\infty} \frac{1}{r}$ diverging even though $\frac{1}{r} \rightarrow 0$ as $r \rightarrow \infty$.

In exploring this, several people's opinions have been that it would not be too much of a stretch to introduce the ratio test (not on A Level syllabus).

BRIEF INTERLUDE : THE RATIO TEST

Consider the series $\sum_{r=1}^{\infty} a_r$, $a_r \neq 0 \ \forall r$.

Let
$$L = \lim_{r \rightarrow \infty} \left| \frac{a_{r+1}}{a_r} \right|$$

I will
prove
this.

→ • if $L < 1$, the series converges absolutely (meaning $\sum_{r=1}^{\infty} |a_r|$ converges
 $\Rightarrow \sum_{r=1}^{\infty} |a_r|$ converges)

- if $L > 1$, the series diverges
- if $L = 1$, either conclusion is possible.

$$\sum_{r=1}^{\infty} a_r, \quad a_r \neq 0 \quad \forall r.$$

Let $L = \lim_{r \rightarrow \infty} \left| \frac{a_{r+1}}{a_r} \right|$ and suppose that $L < 1$.

Then there exists some t such that $L < t < 1$.

$$\text{so } L = \lim_{r \rightarrow \infty} \left| \frac{a_{r+1}}{a_r} \right| < t$$

$$\Rightarrow \exists R \text{ such that } \left| \frac{a_{r+1}}{a_r} \right| < t \quad \forall r \geq R$$

$$\Rightarrow |a_{r+1}| < t |a_r| \quad \forall r \geq R.$$

In particular, $|a_{R+1}| < t |a_R|$

$$|a_{R+2}| < t |a_{R+1}| < t^2 |a_R|$$

$$\dots$$
$$|a_{R+k}| < t |a_{R+k-1}| < t^2 |a_{R+k-2}| < \dots < t^k |a_R|$$
$$\dots$$

$$\text{So } \sum_{r=1}^{\infty} |a_r| = \underbrace{\sum_{r=1}^{R-1} |a_r|}_{\text{finite sum}} + \underbrace{\sum_{r=R}^{\infty} |a_r|}_{\text{geometric series}}$$

This is finite since it is a finite sum

$$\begin{aligned} & |a_R| + |a_{R+1}| + |a_{R+2}| + \dots + |a_{R+k}| + \dots \\ & < |a_R| + t|a_R| + t^2|a_R| + \dots + t^k|a_R| + \dots \\ & = |a_R| (1 + t + t^2 + \dots + t^k + \dots) \end{aligned}$$

converges to $\frac{1}{1-t}$ since $0 \leq L < t < 1$
 $\uparrow$ limit of non-negative values.

$$= \frac{|a_R|}{1-t}, \text{ which is finite, since } t \neq 1.$$

So $\sum_{r=1}^{\infty} |a_r|$ is finite, meaning $\sum_{r=1}^{\infty} |a_r|$ converges $\Rightarrow \sum_{r=1}^{\infty} a_r$ converges.

Generalised Binomial Theorem

How do you explain why $(1+x)^n$ is valid for $|x| < 1$?

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r + \dots, |x| < 1.$$

This is $1 + \sum_{r=1}^{\infty} \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} x^r$

$$L = \lim_{r \rightarrow \infty} \left| \left(\frac{n(n-1)(n-2)\dots(n-r+1)(n-r)}{(r+1)!} x^{r+1} \right) \div \left(\frac{n(n-1)(n-2)\dots(n-r+1)}{r!} x^r \right) \right|$$

$$= \lim_{r \rightarrow \infty} \left| \frac{n-r}{r+1} x \right|$$

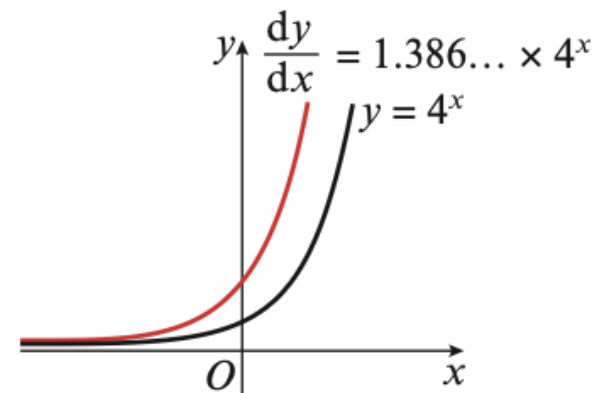
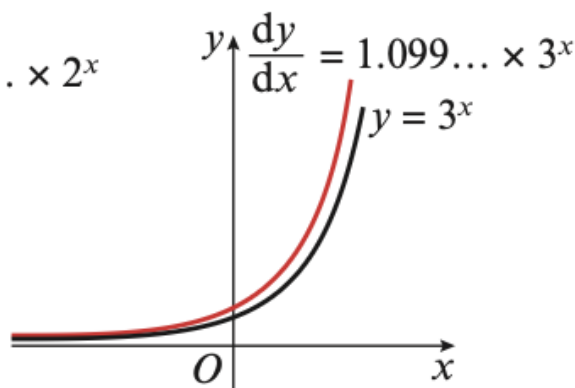
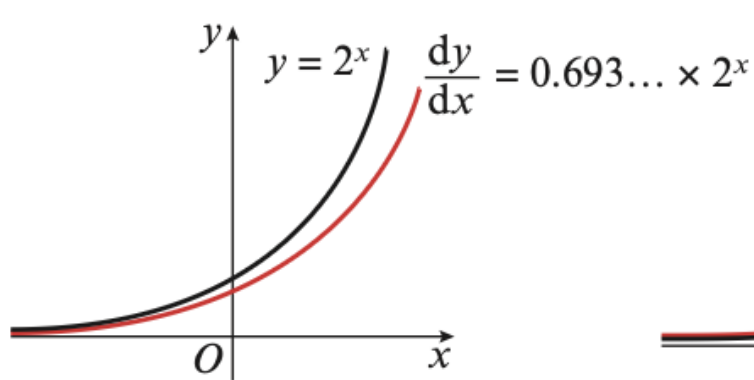
$$= \lim_{r \rightarrow \infty} \left| \frac{\frac{n}{r} - 1}{1 + \frac{1}{r}} \right| |x| = |-1| |x| = |x|$$

So if $L = |x| < 1$ then $\sum_{r=1}^{\infty} \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} x^r$ and hence $1 + \sum_{r=1}^{\infty} \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} x^r$ converge (to the value of $(1+x)^n$).

Exponentials

There is so much beauty contained in the number e , does anyone else feel that we are selling the students short if all we show them is that $y = a^x$ is its own gradient function precisely when $a = e$? Do any of you have ways of introducing more of the properties of e ?

From the EdExcel text book.



$$y = e^x, \quad \frac{dy}{dx} = e^x$$

e is a number between 2 and 3.

// often taught before differentiation...

Exponentials

There is so much beauty contained in the number e , does anyone else feel that we are selling the students short if all we show them is that $y = a^x$ is its own gradient function precisely when $a = e$? Do any of you have ways of introducing more of the properties of e ?

We did not get much time to discuss this, sadly, though I'd love to know people's thoughts. One person shared that she likes to give the following problem when she introduces e :

Place £100 in a bank. How much is there in the bank at the end of a year if the money is left untouched and the compound interest rate is:

100% after 1 year?

50% every 6 months?

25% every 3 months?

...
 $\frac{100}{n}$ % every $\frac{1}{n}$ year?

As $n \rightarrow \infty$, the answer is £100 e , a lovely wow moment, though to understand why would mean proving that $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$.

Another person said how he gets students to explore the function $L(x)$ which is the area under $y = \frac{1}{x}$ between $x=1$ and $x=X$, using rectangles to approximate area. After discovering that $L(x)$ must have logarithmic properties, they find that the base is some number between 2 and 3 ...

Small angle approximations

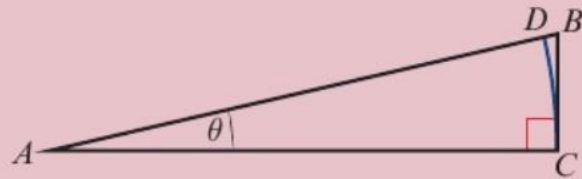
How do you explain these?

■ When θ is small and measured in radians:

- $\sin \theta \approx \theta$
- $\tan \theta \approx \theta$
- $\cos \theta \approx 1 - \frac{\theta^2}{2}$

Challenge

- 1 The diagram shows a right-angled triangle ABC . $\angle BAC = \theta$. An arc, CD , of the circle with centre A and radius AC has been drawn on the diagram in blue.



- a** Write an expression for the arc length CD in terms of AC and θ .
Given that θ is small so that, $AC = AD \approx AB$ and $CD \approx BC$,
- b** deduce that $\sin \theta \approx \theta$ and $\tan \theta \approx \theta$.
- 2 **a** Using the binomial expansion and ignoring terms in x^4 and higher powers of x , find an approximation for $\sqrt{1-x^2}$, $|x| < 1$.
- b** Hence show that for small θ , $\cos \theta \approx 1 - \frac{\theta^2}{2}$. You may assume that $\sin \theta \approx \theta$.

Differentiation

How do you explain **the Chain Rule**? The Product Rule?

Let $F(x) = f(g(x)) = f(u)$ where $u = g(x)$.

and suppose that f and u are differentiable.

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta u} \times \frac{\Delta u}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta u} \times \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x}$$

$\Delta u \rightarrow 0$ as $\Delta x \rightarrow 0$
since $u = g(x)$ is
differentiable, so is continuous

$$= \lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u} \times \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x}$$
$$= \frac{dy}{du} \times \frac{du}{dx}$$

• There is a lot more that
should be said about limits...

• only issue is if $\Delta u = 0$ for
infinitely many values of Δx
as $\Delta x \rightarrow 0$... but this
would correspond to

$$\frac{du}{dx} = 0 \text{ and } \frac{dy}{dx} = 0$$

so the Chain Rule is
still valid.

Differentiation

(The Quotient Rule : $y = \frac{f(x)}{g(x)} = f(x)[g(x)]^{-1}$ and apply Product Rule and Chain Rule).

How do you explain the Chain Rule? **The Product Rule?**

Suppose that $F(x) = f(x)g(x)$ where f and g are both differentiable.

$$F'(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h \rightarrow 0} f(x+h) \times \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} + \lim_{h \rightarrow 0} g(x) \times \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= f(x)g'(x) + g(x)f'(x)$$

One person recommended
a 3 Blue 1 Brown video
on the Product Rule...

Definite Integration

Explaining indefinite integration as the reverse of differentiation (or noitaitnereffid, as I like to call it) feels quite natural, but how do you go about explaining how definite integration leads to giving the signed area between a curve and the x -axis?

I didn't really address this for many years of teaching, and wasn't challenged to do so until last year. Here was my laboured attempt to explain:

Fundamental Theorem of Calculus

FIRST FUNDAMENTAL THEOREM OF CALCULUS!
 Given a continuous function, f ,
 let $F(x) = \int_a^x f(x) dx$.
 Then $F(x)$ is differentiable and $F'(x) = f(x)$.

PROOF

$$F'(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\int_a^{x+h} f(x) dx - \int_a^x f(x) dx}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\int_x^{x+h} f(x) dx + \int_a^x f(x) dx - \int_a^x f(x) dx}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\int_x^{x+h} f(x) dx}{h} \quad (*)$$

Then you apply something as the **MEAN VALUE THEOREM FOR INTEGRALS**, which says that there exists some c , $x < c < x+h$, such that $\int_x^{x+h} f(x) dx = \int_x^{x+h} f(c) dx = f(c)h$.

area as a rectangle of width h and height $f(c)$ for some $x < c < x+h$.

Suppose $f(x) < f(x+h)$. Consider Area $\div h$ (Block area, $f(x+h)$)

area $f(x)h$ $\leq$ $\int_x^{x+h} f(x) dx$ $\leq$ $f(x+h)h$

$\exists c$ s.t. $f(x) < f(c) < f(x+h)$ // True for all $f(c)$ satisfying $f(x) < f(c) < f(x+h)$
 True since f is continuous.
 $\forall C$ s.t. $f(x) < C < f(x+h)$
 $\exists c$ s.t. $f(c) = C$.

so $(*) = \lim_{h \rightarrow 0} \frac{f(c)h}{h} = \lim_{h \rightarrow 0} f(c) = f(x)$ since $x < c < x+h$
 $\Rightarrow c \rightarrow x$ as $h \rightarrow 0$.

SECOND FUNDAMENTAL THEOREM OF CALCULUS

$$\int_a^b F'(x) dx = F(b) - F(a)$$

PROOF

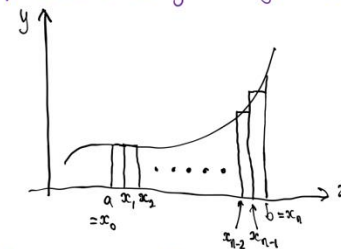
Consider $\int_a^b F'(x) dx$.

Divide the interval $[a, b]$ into n subintervals each of width $\delta x = \frac{b-a}{n}$.

Let $x_0 = a$, $x_1 = a + \frac{b-a}{n}$, $x_2 = a + \frac{2(b-a)}{n}$, ..., $x_n = a + \frac{n(b-a)}{n} = b$.

Let $F'(x) = f(x)$.

Approximate the integral using a Riemann sum.

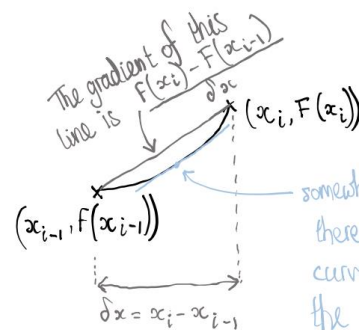


$$\int_a^b f(x) dx \approx \sum_{i=1}^n f(c_i) \delta x \quad \text{where } c_i \text{ can be any point chosen in the interval } (x_{i-1}, x_i)$$

Now also realise that $f(x) = F'(x)$ and so there must be a value d_i in each interval (x_{i-1}, x_i) such that $F'(d_i) = \frac{F(x_i) - F(x_{i-1})}{\delta x}$, so $f(d_i) = \frac{F(x_i) - F(x_{i-1})}{\delta x}$

Definite Integration

Explaining indefinite integration as the reverse of differentiation (or noitaitnereffid, as I like to call it) feels quite natural, but how do you go about explaining how definite integration leads to giving the signed area between a curve and the x-axis?



(If the curve is already a straight line, this is trivial.)

If it isn't, the gradient of the curve either starts as less than the gradient of the chord - but then it has to become greater than the gradient of the chord in order to reach the second point - or it starts as more than the gradient of the chord but then, to reach the second point, it has to become less.)

Since $\int_a^b f(x) dx \approx \sum_{i=1}^n f(c_i) \delta x$ where c_i is ANY value in (x_{i-1}, x_i) , we may as well choose $c_i = d_i$ for each interval.

$$\begin{aligned} \text{So } \int_a^b f(x) dx &\approx \sum_{i=1}^n f(c_i) \delta x \\ \text{becomes } \int_a^b f(x) dx &\approx \sum_{i=1}^n f(d_i) \delta x = \sum_{i=1}^n \frac{F(x_i) - F(x_{i-1})}{\delta x} \delta x = \sum_{i=1}^n F(x_i) - F(x_{i-1}) \\ &= \cancel{F(x_1)} - F(x_0) + \cancel{F(x_2)} - \cancel{F(x_1)} + \cancel{F(x_3)} - \cancel{F(x_2)} + \dots \\ &\quad \dots + \cancel{F(x_{n-1})} - \cancel{F(x_{n-2})} + F(x_n) - \cancel{F(x_{n-1})} \\ &= F(x_n) - F(x_0) = F(b) - F(a) \end{aligned}$$

This is a technique called "Method of Differences" that you learn in A Level Further Maths.

Finally, as $n \rightarrow \infty$, $\delta x \rightarrow 0$ and $\int_a^b f(x) dx \rightarrow \sum_{i=1}^n f(d_i) \delta x$

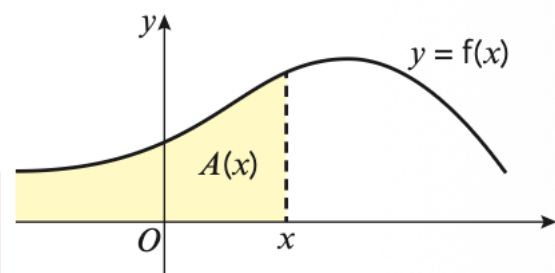
$$\text{so } \int_a^b f(x) dx = \lim_{\delta x \rightarrow 0} \sum_{i=1}^n f(d_i) \delta x = F(b) - F(a) \quad \square$$

(Note that $c_i \in (x_{i-1}, x_i) \rightarrow x_i$ as $\delta x \rightarrow 0$ no matter where exactly c_i is!)

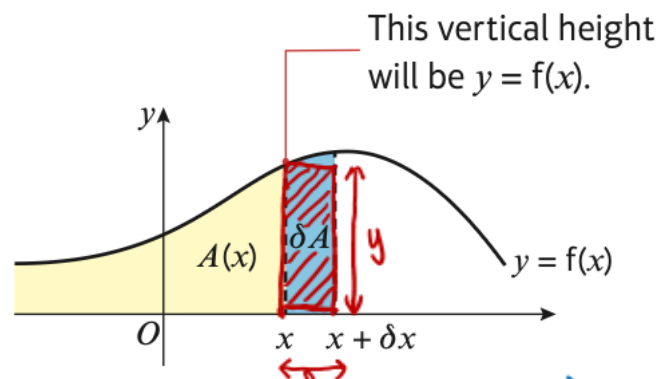
Definite Integration

Explaining indefinite integration as the reverse of differentiation (or noitaitnereffid, as I like to call it) feels quite natural, but how do you go about explaining how definite integration leads to giving the signed area between a curve and the x -axis?

But, credit where credit's due, I hadn't realised that textbooks were trying to address this issue.



STEP ONE: Let $A(x)$ be the signed area between the curve and the x -axis to the left of x .



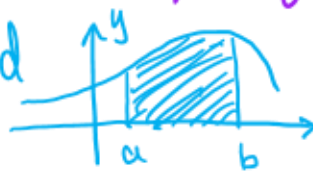
STEP TWO $\delta A = A(x + \delta x) - A(x)$

But also $\delta A \approx y \delta x$ if δx is small.

$$\text{so } A'(x) = \lim_{\delta x \rightarrow 0} \frac{A(x + \delta x) - A(x)}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{y \delta x}{\delta x} = y$$

$$\text{So } A'(x) = y$$

$$\text{and therefore } \int y dx = A(x) + c.$$

and  signed area
 $= A(b) - A(a)$

is what leads to the notation

$$\int_a^b y dx = [A(x)]_a^b$$

Complex roots of a polynomial equation with real coefficients always appear in conjugate pairs.

If $z = a+bi$ and $w = c+di$, $a, b, c, d \in \mathbb{R}$.

$$\text{then } zw = (a+bi)(c+di) = ac + adi + bci + bdi^2 = ac - bd + (ad + bc)i$$

$$\begin{aligned} \text{but } z^* w^* &= (a-bi)(c-di) = ac - adi - bci + bdi^2 = ac - bd - (ad + bc)i \\ &= (zw)^* \end{aligned}$$

Complex roots of a polynomial equation with real coefficients always appear in conjugate pairs.

$$(zw)^* = z^* w^*$$

$$\text{In particular, } (z^2)^* = (z^*)^2$$

$$\text{and } (z^n)^* = (z^*)^n \text{ for any positive integer, } n$$

Complex roots of a polynomial equation with real coefficients always appear in conjugate pairs.

$$(zw)^* = z^* w^*$$

$$(z^n)^* = (z^*)^n \text{ for any } n \in \mathbb{Z}^+$$

$$\begin{aligned} \text{If } k, a, b \in \mathbb{R}, \quad (k(a+bi))^* &= (ka+kbi)^* = ka-kbi \\ &= k(a-bi) = k(a+bi)^* \\ \text{so } (kz)^* &= k z^* \text{ if } k \in \mathbb{R}. \end{aligned}$$

Complex roots of a polynomial equation with real coefficients always appear in conjugate pairs.

$$(zw)^* = z^* w^*$$

$$(z^n)^* = (z^*)^n \text{ for any } n \in \mathbb{Z}^+$$

$$(kz)^* = k z^* \text{ for any } k \in \mathbb{R}$$

$$\begin{aligned}(z+w)^* &= (a+bi+c+di)^* = ((a+c)+(b+d)i)^* \\&= (a+c)-(b+d)i \\&= a-bi+c-di \\&= z^* + w^*\end{aligned}$$

Complex roots of a polynomial equation with real coefficients always appear in conjugate pairs.

$$(zw)^* = z^* w^* \quad (1)$$

$$(z^n)^* = (z^*)^n \text{ for any } n \in \mathbb{Z}^+ \quad (2)$$

$$(kz)^* = k z^* \text{ for any } k \in \mathbb{R} \quad (3)$$

$$(z+w)^* = z^* + w^* \quad (4)$$

Suppose that $z = z_1 \in \mathbb{C} \setminus \mathbb{R}$ is a root of the polynomial equation

$$p_0 + p_1 z + p_2 z^2 + p_3 z^3 + \dots + p_{n-1} z^{n-1} + p_n z^n = 0, \quad p_i \in \mathbb{R} \quad \forall i$$

$$\text{so that } p_0 + p_1 z_1 + p_2 z_1^2 + p_3 z_1^3 + \dots + p_{n-1} z_1^{n-1} + p_n z_1^n = 0$$

$$\begin{aligned} \text{Then } & p_0 + p_1 z_1^* + p_2 (z_1^*)^2 + p_3 (z_1^*)^3 + \dots + p_{n-1} (z_1^*)^{n-1} + p_n (z_1^*)^n \\ &= p_0 + p_1 z_1^* + p_2 (z_1^2)^* + p_3 (z_1^3)^* + \dots + p_{n-1} (z_1^{n-1})^* + p_n (z_1^n)^* \quad \text{by (2)} \\ &= p_0 + (p_1 z_1)^* + (p_2 z_1^2)^* + (p_3 z_1^3)^* + \dots + (p_{n-1} z_1^{n-1})^* + (p_n z_1^n)^* \quad \text{by (3)} \\ &= (p_0 + p_1 z_1 + p_2 z_1^2 + p_3 z_1^3 + \dots + p_{n-1} z_1^{n-1} + p_n z_1^n)^* \quad \text{by (4)} \\ &= 0^* = 0 \end{aligned}$$

So if $z = z_1$ is a root then so is $z = z_1^*$

Invariant lines.

You are taught to find invariant lines of the form $y = mx + c$, but it should be addressed that not all straight lines can be expressed in this form... so might there be others?

The only lines that cannot be expressed in the form $y = mx + c$ are vertical lines, and these take the form $x = a$ for some constant, a .

Let (a, y) be a general point on the line $x = a$.

Suppose that $x = a$ is an invariant line of the transformation given by the matrix $\begin{pmatrix} m & n \\ p & q \end{pmatrix}$.

Then $\begin{pmatrix} m & n \\ p & q \end{pmatrix} \begin{pmatrix} a \\ y \end{pmatrix} = \begin{pmatrix} ma + ny \\ pa + qy \end{pmatrix}$ and so $(ma + ny, pa + qy)$ must also lie on $x = a$.

For this to happen, $ma + ny$ must equal a as y varies through all real numbers.

For $ma + ny$ simply to be a constant, it is necessary that $n = 0$.

Then $ma = a$ requires either $m = 1$ or $a = 0$.

Invariant lines.

You are taught to find invariant lines of the form $y = mx + c$, but it should be addressed that not all straight lines can be expressed in this form... so might there be others?

$m=1, n=0$ gives:

$$\begin{pmatrix} 1 & 0 \\ p & q \end{pmatrix} \begin{pmatrix} a \\ y \end{pmatrix} = \begin{pmatrix} a \\ pa+qy \end{pmatrix} \quad \text{so } (a, y) \text{ is mapped to a point also on } x=a \text{ (any } a \in \mathbb{R})$$

and $x=a$ is an invariant line.

$n=0$ and $a=0$ gives:

$$\begin{pmatrix} m & 0 \\ p & q \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ qy \end{pmatrix} \quad \text{so } (0, y) \text{ is mapped to a point also on } x=0$$

and $x=0$, the y -axis is an invariant line.

So any matrix of the form $\begin{pmatrix} m & 0 \\ p & q \end{pmatrix}$ has the y -axis as an invariant line, and if $m=1$ then every vertical line is an invariant line.

If the matrix is not of the form $\begin{pmatrix} m & 0 \\ p & q \end{pmatrix}$ then there are no vertical invariant lines.

Second order differential equations.

When you show how the general solution of the of $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$, when the auxiliary equation has complex roots, $\alpha \pm i\beta$, is $y = e^{\alpha x} (C \cos \beta x + D \sin \beta x)$, how do you explain the appearance/disappearance of i ? After all, in the case of Simple Harmonic Motion, for example, the solution is entirely based in the 'real world'. Talking of which, do you have a concrete way of explaining the three cases for damped SHM?

$$y = Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} \\ = Ae^{\alpha x} e^{i\beta x} + Be^{\alpha x} e^{-i\beta x}$$

One 'leap of faith' for the student to just accept is that A and B can be complex numbers.

$$= Ae^{\alpha x} (\cos \beta x + i \sin \beta x) + Be^{\alpha x} (\cos \beta x - i \sin \beta x) \\ = e^{\alpha x} ((A+B) \cos \beta x + (A-B)i \sin \beta x) = e^{\alpha x} (C \cos \beta x + D \sin \beta x)$$

where $C = A+B$ and $D = (A-B)i$

for this to be a real number solution would require

$$A+B \in \mathbb{R} \quad \text{and} \quad (A-B)i \in \mathbb{R}$$

So is this what is going on?!

$A+B \in \mathbb{R} \Rightarrow \text{Im}(A) = -\text{Im}(B)$ and $(A-B)i \in \mathbb{R} \Rightarrow \text{Re}(A) = \text{Re}(B)$ so A and B would have to be a complex conjugate pair.

Second order differential equations.

Finally, for non-homogeneous second order differential equations how do you justify using the trial function $y = xf(x)$ when $f(x)$ is already part of the complementary function? Come to think of it, **how do you explain why the general solution of a homogeneous second order differential equation is $y = Ae^{\lambda_1 x} + Bxe^{\lambda_1 x}$ when the auxiliary equation has repeated root λ_1 ?**

Suppose that $a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$, and that $a\lambda^2 + b\lambda + c = 0$ has repeated root $\lambda = \lambda_1$.

$$y = Ae^{\lambda_1 x} + Bxe^{\lambda_1 x}$$

$$\Rightarrow \frac{dy}{dx} = A\lambda_1 e^{\lambda_1 x} + Be^{\lambda_1 x} + B\lambda_1 x e^{\lambda_1 x}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = A\lambda_1^2 e^{\lambda_1 x} + B\lambda_1 e^{\lambda_1 x} + B\lambda_1 e^{\lambda_1 x} + B\lambda_1^2 x e^{\lambda_1 x}$$

$$\begin{aligned} \text{so } a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy &= a \left(A\lambda_1^2 e^{\lambda_1 x} + B\lambda_1 e^{\lambda_1 x} + B\lambda_1 e^{\lambda_1 x} + B\lambda_1^2 x e^{\lambda_1 x} \right) \\ &\quad + b \left(A\lambda_1 e^{\lambda_1 x} + Be^{\lambda_1 x} + B\lambda_1 x e^{\lambda_1 x} \right) \\ &\quad + c \left(Ae^{\lambda_1 x} + Bxe^{\lambda_1 x} \right) \end{aligned}$$

Second order differential equations.

Finally, for non-homogeneous second order differential equations how do you justify using the trial function $y = xf(x)$ when $f(x)$ is already part of the complementary function? Come to think of it, **how do you explain why the general solution of a homogeneous second order differential equation is $y = Ae^{\lambda_1 x} + Bxe^{\lambda_1 x}$ when the auxiliary equation has repeated root λ_1 ?**

Suppose that $a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$, and that $a\lambda^2 + b\lambda + c = 0$ has repeated root $\lambda = \lambda_1$.

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = e^{\lambda_1 x} \left(\underbrace{aA\lambda_1^2 + 2aB\lambda_1 + bA\lambda_1 + bB + cA}_{=0} \right) + x e^{\lambda_1 x} \left(\underbrace{aB\lambda_1^2 + bB\lambda_1 + cB}_{=0} \right)$$

But $\lambda = \lambda_1$ is a root of $a\lambda^2 + b\lambda + c = 0 \Rightarrow \underbrace{a\lambda_1^2 + b\lambda_1 + c}_{=0} = 0 \Rightarrow aB\lambda_1^2 + bB\lambda_1 + cB = \underline{\underline{0}}$

$$aA\lambda_1^2 + 2aB\lambda_1 + bA\lambda_1 + bB + cA = A \left(\underbrace{a\lambda_1^2 + b\lambda_1 + c}_{=0} \right) + B \left(\underbrace{2a\lambda_1 + b}_{=0} \right)$$

But $\lambda = \lambda_1$ is a repeated root
so $\frac{d}{d\lambda}(a\lambda^2 + b\lambda + c) \Big|_{\lambda=\lambda_1} = 0 \Rightarrow 2a\lambda_1 + b = \underline{\underline{0}}$

$$\text{so } a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

Second order differential equations.

How do you explain that the most general solution to a second order differential equation has two unknowns? It is clear when the equation is of the form $\frac{d^2y}{dx^2} = f(x)$, but less obvious when it is of the form $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$.

We didn't get time to address this in the session but someone did share at the end that they show how a substitution can reduce a second order differential equation to something of the form $\frac{d^2y}{dx^2} = f(x)$, which does clearly have two unknowns.

Second order differential equations.

8.3 Damped and forced harmonic motion

You can refine the model for simple harmonic motion by adding an additional force which is proportional to the velocity of the particle. When this force acts so as to slow the particle down it is known as a **damping force**, and the motion of the particle is known as **damped harmonic motion**.

■ For a particle moving with damped harmonic motion

$$\frac{d^2x}{dt^2} + k \frac{dx}{dt} + \omega^2 x = 0$$

where x is the displacement from a fixed point at time t , and k and ω^2 are positive constants.

There are three separate cases corresponding to the auxiliary equation having distinct real, equal or complex roots.

When $k^2 > 4\omega^2$ there are two distinct real roots for the auxiliary equation. This is known as **heavy damping**. In this case there will be no oscillations performed as the resistive force is large compared with the restoring force.

When $k^2 = 4\omega^2$ the auxiliary equation has equal roots. This is known as **critical damping**. Again there will be no oscillations performed.

When $k^2 < 4\omega^2$ the auxiliary equation has complex roots. This is known as **light damping** and is the only case where oscillations are seen. The amplitude of the oscillations will decrease exponentially over time.

For heavy and critical damping the exact nature of the motion will depend on the initial conditions given.

For light damping the period of the observed oscillations can be calculated.

Notation You could also write this as $\ddot{x} + k\dot{x} + \omega^2 x = 0$.

Links This is an example of a second-order homogeneous differential equation. You can solve equations like this by considering the auxiliary equation. [← Section 7.2](#)

We didn't get the chance to discuss this but I would like to have more than the basic understanding I currently have. Anyone want to discuss?!

Maclaurin Series

Do any of you justify the 'valid for' part of the Maclaurin Series that students are told to just use blindly from the formula book? If so, how do you do it?

The Further Maths EdExcel textbook does refer to the ratio test, but only as a Challenge.

Challenge

The **ratio test** is a sufficient condition for the convergence of an infinite series. It says that a series $\sum_{r=1}^{\infty} a_r$ converges if $\lim_{r \rightarrow \infty} \left| \frac{a_{r+1}}{a_r} \right| < 1$, and diverges if $\lim_{r \rightarrow \infty} \left| \frac{a_{r+1}}{a_r} \right| > 1$.

Use the ratio test to show that

- a** the Maclaurin series expansion of e^x converges for all $x \in \mathbb{R}$
- b** the Maclaurin series expansion of $\ln(1+x)$ converges for $-1 < x < 1$, and diverges for $x > 1$.

I have used the ratio test to derive the 'valid for' statements that accompany the Maclaurin Series.
I was surprised how straightforward this was ... why don't we do this?!

Configuration of planes when there is not a unique intersection point.

Is there a strategy behind what you teach? Are you able to get your students to understand why your strategy works?

Easiest to show an example!

- a Determine the values of the real constants a and b for which there are infinitely many solutions to the simultaneous equations

$$2x + 3y + z = 6 \quad (1)$$

$$-x + y + 2z = 7 \quad (2)$$

$$ax + y + 4z = b \quad (3)$$

(6 marks)

- b Give a geometric interpretation of the three planes formed by these equations.

(1 mark)

My strategy: eliminate the same variable from two pairs of equations:

$$2 \times (1) - (2): \quad 5x + 5y = 5 \\ \Rightarrow x + y = 1 \quad (4)$$

$$4 \times (1) - (3): \quad (8-a)x + 11y = 24-b \quad (5)$$

(4) and (5) have a unique solution if and only if the left-hand sides are not multiples of each other.
Unique solution $\Leftrightarrow 8-a \neq 11 \Leftrightarrow a \neq -3$.

Configuration of planes when there is not a unique intersection point.

Is there a strategy behind what you teach? Are you able to get your students to understand why your strategy works?

Easiest to show an example!

- a Determine the values of the real constants a and b for which there are infinitely many solutions to the simultaneous equations

$$2x + 3y + z = 6 \quad (1)$$

$$-x + y + 2z = 7 \quad (2)$$

$$ax + y + 4z = b \quad (3)$$

(6 marks)

- b Give a geometric interpretation of the three planes formed by these equations.

(1 mark)

There is not a unique solution $\Leftrightarrow a = -3$.

$$(4) : x + y = 1$$

$$\text{so } 11x + 11y = 11$$

$$(5) : (8-a)x + 11y = 24-b, \text{ so } 11x + 11y = 24-b.$$

These are consistent iff $11 = 24 - b \Leftrightarrow b = 13$

So there are infinitely many solutions if $b = 13$
(and no solutions if $b \neq 13$)

Configuration of planes when there is not a unique intersection point.

Is there a strategy behind what you teach? Are you able to get your students to understand why your strategy works?

Easiest to show an example!

- a Determine the values of the real constants a and b for which there are infinitely many solutions to the simultaneous equations

$$2x + 3y + z = 6 \quad (1)$$

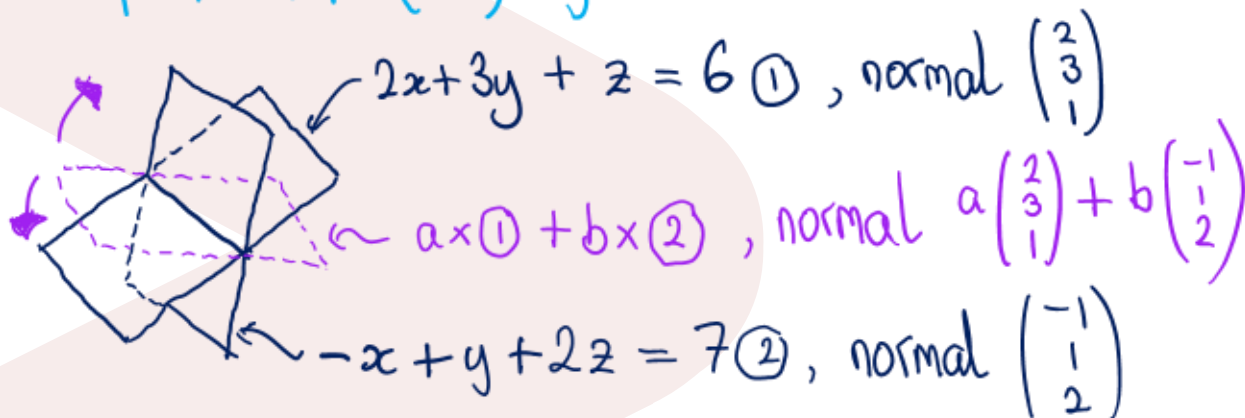
$$-x + y + 2z = 7 \quad (2)$$

$$ax + y + 4z = b \quad (3)$$

- b Give a geometric interpretation of the three planes formed by these equations.

b) Answer: a sheaf of planes

The "Oops-no-Wi-Fi (Non-)Geogebra Demonstration":



In my solution, I did

$2 \times (1) - (2)$ to eliminate $z \dots$

This found the equation of the plane that contains the line of intersection of (1) and (2) and has normal of the form

$$\begin{pmatrix} p \\ q \\ 0 \end{pmatrix}$$

(6 marks)

(1 mark)

then I did $4 \times (1) - (3)$ to eliminate $z \dots$

This found the equation of the plane that contains the line of intersection of (1) and (3) and has normal also of the form

$$\begin{pmatrix} p \\ q \\ 0 \end{pmatrix}$$

These planes are parallel if the system has no unique solution.

Configuration of planes when there is not a unique intersection point.

Is there a strategy behind what you teach? Are you able to get your students to understand why your strategy works?

Easiest to show an example!

- a Determine the values of the real constants a and b for which there are infinitely many solutions to the simultaneous equations

$$2x + 3y + z = 6 \quad \textcircled{1}$$

$$-x + y + 2z = 7 \quad \textcircled{2}$$

$$ax + y + 4z = b \quad \textcircled{3}$$

(6 marks)

- b Give a geometric interpretation of the three planes formed by these equations.

(1 mark)

From my working earlier

$$11x + 11y = 11$$

$$11x + 11y = 24 - b$$

}

A pair of parallel planes (each have normal $\begin{pmatrix} 11 \\ 11 \\ 0 \end{pmatrix}$)
If they are the same plane, the simultaneous equations have infinitely many solutions


$$2\textcircled{1} - \textcircled{2} = 4\textcircled{1} - \textcircled{3}$$

Configuration of planes when there is not a unique intersection point.

Is there a strategy behind what you teach? Are you able to get your students to understand why your strategy works?

Easiest to show an example!

- a Determine the values of the real constants a and b for which there are infinitely many solutions to the simultaneous equations

$$2x + 3y + z = 6$$

$$-x + y + 2z = 7$$

$$ax + y + 4z = b$$

(6 marks)

- b Give a geometric interpretation of the three planes formed by these equations.

(1 mark)

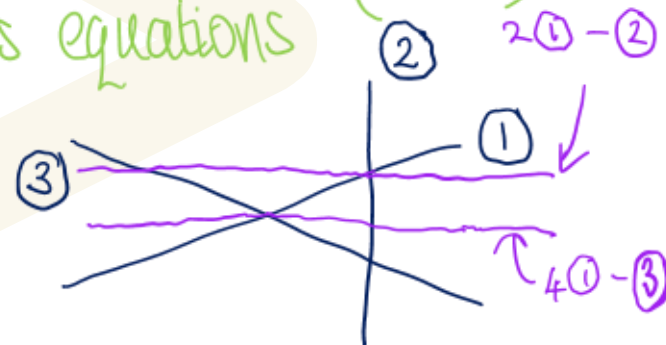
From my working earlier

$$11x + 11y = 11$$

$$11x + 11y = 24 - b$$

}

A pair of parallel planes (each have normal $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$)
if they are not the same plane ($b \neq 13$)
then the simultaneous equations
have no solutions.



What is the integral, $\int \frac{1}{2x} dx$?

EITHER $\int \frac{1}{2x} dx = \frac{1}{2} \int \frac{2}{2x} dx = \frac{1}{2} \ln|2x| + C$

OR $\int \frac{1}{2x} dx = \frac{1}{2} \int \frac{1}{x} dx = \frac{1}{2} \ln|x| + C$

Neither are wrong!! It's just that the C 's are different!

$$\frac{1}{2} \ln|2x| + C = \frac{1}{2} (\ln(2) + \ln|x|) + C$$

$$= \frac{1}{2} \ln(2) + \frac{1}{2} \ln|x| + C$$

$$= \frac{1}{2} \ln|x| + \boxed{C + \frac{1}{2} \ln(2)}$$

constant of integration

From the formula book for EdExcel:

Integration (+ constant; $a > 0$ where relevant)

$f(x)$	$\int f(x) dx$
$\frac{1}{\sqrt{x^2 - a^2}}$	$\operatorname{arcosh}\left(\frac{x}{a}\right), \ln\{x + \sqrt{x^2 - a^2}\} \quad (x > a)$
$\frac{1}{\sqrt{a^2 + x^2}}$	$\operatorname{arsinh}\left(\frac{x}{a}\right), \ln\{x + \sqrt{x^2 + a^2}\}$

$$\begin{aligned}\text{In fact, } \operatorname{arcosh}\left(\frac{x}{a}\right) &= \ln\left\{\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 - 1}\right\} \\ &= \ln\left\{\frac{x}{a} + \sqrt{\frac{x^2 - a^2}{a^2}}\right\} \\ &= \ln\left\{\frac{x + \sqrt{x^2 - a^2}}{a}\right\} \\ &= \ln\{x + \sqrt{x^2 - a^2}\} - \ln\{a\}\end{aligned}$$

$$\begin{aligned}\text{and, similarly, } \operatorname{arsinh}\left(\frac{x}{a}\right) &= \ln\left\{\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 + 1}\right\} \\ &= \dots = \ln\{x + \sqrt{x^2 + a^2}\} - \ln\{a\}.\end{aligned}$$

This isn't wrong but it is misleading.
Each year students use this to wrongly
conclude that

$$\begin{aligned}\operatorname{arcosh}\left(\frac{x}{a}\right) &= \ln\{x + \sqrt{x^2 - a^2}\} \\ \text{and } \operatorname{arsinh}\left(\frac{x}{a}\right) &= \ln\{x + \sqrt{x^2 + a^2}\}\end{aligned}$$


So the formula book is not wrong
but the two equivalent correct answers
in each case are not equal and differ
by a constant (of integration), $\ln\{a\}$.

MY STUDENT'S WORK

15. The number of mice on an island is being monitored.

When monitoring began there were 4000 mice on the island.

In a simple model, the number of mice x , in thousands, is modelled by the equation

$$x = \frac{k(3t+2)}{4t+5}$$

where k is a constant and t is the number of years after monitoring began.

(a) Find a complete equation for this model.

(b) Hence calculate the long-term population of mice predicted by this model.

In a second model, the number of mice x , in thousands, is modelled by the differential equation

$$4 \frac{dx}{dt} = 6x - x^2$$

where t is the number of years after monitoring began.

(c) Write $\frac{4}{6x-x^2}$ in partial fraction form.

(d) Solve the differential equation with the given initial condition to show that

$$x = \frac{12e^{\frac{3}{2}t}}{1+2e^{\frac{3}{2}t}}$$

(e) Hence find the long-term population of mice predicted by this **second** model.

a) sub. $t=0$,
 $\frac{k(2)}{5} = 4$

$$2k = 20$$

$$k = 10$$

$$x = \frac{10(3t+2)}{4t+5}$$

b) 7500



All good so far...

15. The number of mice on an island is being monitored.

When monitoring began there were 4000 mice on the island.

In a simple model, the number of mice x , in thousands, is modelled by the equation

$$x = \frac{k(3t+2)}{4t+5}$$

where k is a constant and t is the number of years after monitoring began.

(a) Find a complete equation for this model.

(b) Hence calculate the long-term population of mice predicted by this model.

In a second model, the number of mice x , in thousands, is modelled by the differential equation

$$4 \frac{dx}{dt} = 6x - x^2$$

where t is the number of years after monitoring began.

(c) Write $\frac{4}{6x-x^2}$ in partial fraction form.

(d) Solve the differential equation with the given initial condition to show that

$$x = \frac{12e^{\frac{3}{2}t}}{1+2e^{\frac{3}{2}t}}$$

(e) Hence find the long-term population of mice predicted by this **second** model.

a) sub. $t=0$,
 $\frac{k(2)}{5} = 4$

$$2k = 20$$

$$k = 10$$

$$x = \frac{10(3t+2)}{4t+5}$$

b) 7500

MY STUDENT'S WORK

Question 15 continued

c) $6x - x^2 = x(6-x)$

$$A(6-x) + Bx = 4$$

$$B-A=0$$

$$6A=4$$

$$A = \frac{2}{3}$$

$$\frac{\frac{2}{3}}{x} + \frac{\frac{2}{3}}{6-x}$$

$$= \frac{2}{3x} - \frac{2}{3x-18}$$

d) $\int \frac{2}{3x} dx - \int \frac{2}{3x-18} dx = \int 1 dt$

$$\frac{2}{3} \int x^{-1} dx - \frac{2}{3} \int (x-6)^{-1} dx = t$$

$$\frac{2}{3} \ln|x| - \frac{2}{3} \ln|x-6| + C = t$$

$$\frac{2}{3} \ln|4| - \frac{2}{3} \ln|-2| + C = 0$$

$$\frac{2}{3} \ln(2) + C = 0$$

$$C = -\frac{2}{3} \ln(2) < \frac{2}{3} (\ln(x) - \ln(x-6) - \ln(2)) = t$$

$$\frac{2}{3} \left(\ln \left(\frac{x}{2x-12} \right) \right) = t$$

$$\ln \left(\frac{x}{2x-12} \right) = \frac{3}{2} t$$

$$\frac{x}{2x-12} = e^{\frac{3}{2} t}$$

$$x = 2x e^{\frac{3}{2} t} - 12 e^{\frac{3}{2} t}$$

$$x(1 - 2e^{\frac{3}{2} t}) = -12 e^{\frac{3}{2} t}$$

$$x = \frac{12 e^{\frac{3}{2} t}}{2e^{\frac{3}{2} t} - 1}$$

e) 6000

"Correct but completely unnecessary show-off move," I initially thought.

$\frac{\frac{2}{3}}{6-x}$ is the same as $-\frac{2}{3(x-6)}$, and perhaps this was done not to show off but to integrate not with $-x$ but x in the denominator... not actually a crazy idea on certain occasions but here it sends the student down a completely different path to the one imagined by the question setter.

I followed this working very carefully and, not finding any mistakes, I gave the answer full marks 'n!!

It was only the honesty of the student after the exam that drew me to this:

$$x = \frac{12e^{\frac{3}{2}t}}{1+2e^{\frac{3}{2}t}}?$$

MY STUDENT'S WORK

Question 15 continued

c) $6x - x^2 = x(6-x)$

$$A(6-x) + Bx = 4$$

$$B-A=0$$

$$6A=4$$

$$A = \frac{2}{3}$$

$$\frac{\frac{2}{3}}{x} + \frac{\frac{2}{3}}{6-x}$$

$$= \frac{2}{3x} + \frac{2}{3(6-x)}$$

d) $\int \frac{2}{3x} dx - \int \frac{2}{3(6-x)} dx = \int 1 dt$

$$\frac{2}{3} \int x^{-1} dx - \frac{2}{3} \int (x-6)^{-1} dx = t$$

$$\frac{2}{3} \ln|x| - \frac{2}{3} \ln|x-6| + C = t$$

$$\frac{2}{3} \ln|4| - \frac{2}{3} \ln|-2| + C = 0$$

$$\frac{2}{3} \ln(2) + C = 0$$

$$C = -\frac{2}{3} \ln(2)$$

$$\frac{2}{3} \left(\ln\left(\frac{x}{2(6-x)}\right) \right) = t$$

$$\ln\left(\frac{x}{2(6-x)}\right) = \frac{3}{2}t$$

$$\frac{x}{2(6-x)} = e^{\frac{3}{2}t}$$

$$x = 2xe^{\frac{3}{2}t} - 12e^{\frac{3}{2}t}$$

$$x(1-2e^{\frac{3}{2}t}) = -12e^{\frac{3}{2}t}$$

$$x = \frac{12e^{\frac{3}{2}t}}{2e^{\frac{3}{2}t}-1}$$

not what asked for in the question!

e) 6000

THE EXPECTED APPROACH.

using $\frac{2}{3x} + \frac{2}{3(6-x)}$ from c)

d) $\int \frac{4}{6x-x^2} dx = \int 1 dt$

$$\int \left(\frac{2}{3x} + \frac{2}{3(6-x)} \right) dx = \int 1 dt$$

$$\frac{2}{3} \int \left(\frac{1}{x} + \frac{1}{6-x} \right) dx = \int 1 dt$$

$$\frac{2}{3} (\ln|x| - \ln|6-x|) = t + C$$

$x=4$ when $t=0$: $\frac{2}{3} (\ln 4 - \ln 2) = C$, so $C = \frac{2}{3} \ln 2$.

so $\frac{2}{3} \ln \left| \frac{x}{6-x} \right| = t + \frac{2}{3} \ln 2$

$\times \frac{3}{2}$: $\ln \left| \frac{x}{6-x} \right| = \frac{3}{2}t + \ln 2$

$$\frac{3}{2}t = \ln \left| \frac{x}{6-x} \right| - \ln 2 = \ln \left| \frac{x}{2(6-x)} \right|$$

$$e^{\frac{3}{2}t} = \frac{x}{12-2x} \Rightarrow e^{\frac{3}{2}t} (12-2x) = x$$

$$\Rightarrow 12e^{\frac{3}{2}t} = x(1+2e^{\frac{3}{2}t}) \Rightarrow x = \frac{12e^{\frac{3}{2}t}}{1+2e^{\frac{3}{2}t}} \text{ as required.}$$

MY STUDENT'S WORK

$$\frac{2}{3} \left(\ln \left| \frac{x}{2x-12} \right| \right) = t$$

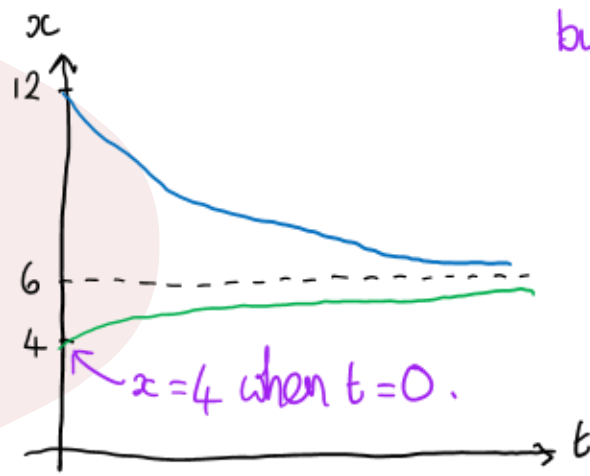
$$\ln \left| \frac{x}{2x-12} \right| = \frac{3}{2}t$$

$$\Rightarrow \left| \frac{x}{2x-12} \right| = e^{\frac{3}{2}t}$$



$$\frac{x}{12-2x} = \pm e^{\frac{3}{2}t}$$

leads to $x = \frac{12e^{\frac{3}{2}t}}{2e^{\frac{3}{2}t} - 1}$
 $(= 6 + \frac{6}{2e^{\frac{3}{2}t} - 1})$



THE EXPECTED APPROACH

$$\frac{3}{2}t = \ln \left| \frac{x}{6-x} \right| - \ln 2 = \ln \left| \frac{x}{2(6-x)} \right|$$

$$\ln \left| \frac{x}{12-2x} \right| = \frac{3}{2}t$$

$$\Rightarrow \left| \frac{x}{12-2x} \right| = e^{\frac{3}{2}t}$$



Both the + and the - correspond to solutions of $4 \frac{dx}{dt} = 6x - x^2$ but only one of these satisfies $x=4$ when $t=0$.

$$\frac{x}{12-2x} = e^{\frac{3}{2}t}$$

leads to $x = \frac{12e^{\frac{3}{2}t}}{2e^{\frac{3}{2}t} + 1}$
 $(= 6 - \frac{6}{2e^{\frac{3}{2}t} + 1})$

Use the substitution $u^2 = \sin x$ to evaluate the definite integral $\int_0^\pi \sqrt{1 - \sin x} \, dx$.

$$u^2 = \sin x \Rightarrow 2u \frac{du}{dx} = \cos x$$

NB. $\sin x \geq 0$

$$\text{on } 0 \leq x \leq \pi \Rightarrow \frac{2u}{\cos x} du = dx$$

$$\text{But } \cos x = \pm \sqrt{\cos^2 x} = \pm \sqrt{1 - \sin^2 x} = \pm \sqrt{1 - (u^2)^2} = \pm \sqrt{1 - u^4}$$

$$\cos x = \begin{cases} \sqrt{1 - u^4} & \text{on } 0 \leq x \leq \frac{\pi}{2} \text{ (when } \cos x \geq 0) \\ -\sqrt{1 - u^4} & \text{on } \frac{\pi}{2} \leq x \leq \pi \text{ (when } \cos x \leq 0) \end{cases}$$

So we need to split up the Integral!

$$\int_0^\pi \sqrt{1 - \sin x} \, dx = \int_0^{\pi/2} \sqrt{1 - \sin x} \, dx + \int_{\pi/2}^\pi \sqrt{1 - \sin x} \, dx$$

$$= \int_0^1 \sqrt{1 - u^2} \cdot \frac{2u}{\sqrt{1 - u^4}} du + \int_1^0 \sqrt{1 - u^2} \cdot \frac{2u}{-\sqrt{1 - u^4}} du$$

$$x = \frac{\pi}{2} \Rightarrow u^2 = 1 \Rightarrow u = \pm 1$$

$$x = 0 \Rightarrow u^2 = 0$$

$$x = \pi \Rightarrow u^2 = 0$$

Use the substitution $u^2 = \sin x$ to evaluate the definite integral $\int_0^\pi \sqrt{1 - \sin x} \, dx$.

$$\begin{aligned} & \int_0^1 \frac{2u\sqrt{1-u^2}}{\sqrt{1-u^4}} du - \int_1^0 \frac{2u\sqrt{1-u^2}}{\sqrt{1-u^4}} du \\ &= \int_0^1 \frac{2u}{\sqrt{1+u^2}} du + \int_0^1 \frac{2u}{\sqrt{1+u^2}} du \\ &= 2 \int_0^1 \frac{2u}{\sqrt{1+u^2}} du \\ &= 2 \left[2(1+u^2)^{1/2} \right]_0^1 = 2 \left[2(1+1^2)^{1/2} - 2(1+0^2)^{1/2} \right] \\ &= 2(2\sqrt{2} - 2) = 4(\sqrt{2} - 1) \end{aligned}$$

NB. if 1 is replaced everywhere with -1, you get the same answer.

(e.g. as x increases from 0 to $\frac{\pi}{2}$, $u^2 = \sin x$ increases from 0 to 1, but this can also be achieved by u decreasing from 0 to -1).



CREDITS:

MELONY

EGOR

MIKKII

GO FURTHER,
COME RUN WITH
THE SWIFT